JOURNAL OF MULTIVARIATE ANALYSIS 10, 95–106 (1980)

Convergence of Weighted Sums of Random Elements in $D[0, 1]$

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Communicated by E. Lukacs

Convergence in probability for Toeplitz weighted sums is obtained for convex tight random elements in $D[0, 1]$ under pointwise conditions. The almost sure convergence of the weighted sums is proved for independent, convex tight random elements and for independent, identically distributed random elements. Special techniques and concepts are developed in order to obtain these results in the Skorohod topology of $D[0, 1]$.

1. INTRODUCTION

The use of the space $D[0, 1]$ with the Skorohod topology has proved to be very beneficial not only to statisticians in developing statistical theory but also for many statistical applications. The basic framework and many results for probability structures in $D[0, 1]$ are given by Billingsley (1968). However, the peculiar structure and properties of $D[0, 1]$ under the Skorohod topology have impeded development of convergent results for weighted sums of random elements. For example, the results for separable Banach spaces do not readily apply. However, by introducing particular techniques some laws of large numbers were obtained by Ranga Rao (1963) and Daffer and Taylor (1979).

Convergence of weighted sums of random elements results are obtained for $D[0, 1]$ in this paper using convexity, compactness, and truncation techniques. Specifically, convergence in probability for Toeplitz weighted sums is obtained under pointwise conditions for a large class of random elements in $D[0, 1]$.

Received April 10, 1979.

AMS 1970 subject classifications: Primary 60B10.

Key words and Phrases: convergence in probability, weighted sums, Skorohod topology.

0047-259X/80/010095-12$02.00/0

Almost sure convergence of the weighted sums is obtained for independent convex tight random elements and for independent, identically distributed random elements in Section 4.

2. Preliminaries

Let $D[0, 1](= D)$ denote the space of real-valued functions on $[0, 1]$ which are right-continuous and possess left-hand limits. Let $\| x \|_\infty = \sup_{0 \leqslant t \leqslant 1} | x(t)|$ denote the uniform norm on D. The Skorohod metric d is defined for $x, y \in D$ by

$$d(x, y) = \inf\{\epsilon : \| x \circ \lambda - y \|_\infty \leqslant \epsilon \;\&\; \| \lambda - \lambda_0 \|_\infty \leqslant \epsilon \text{ with } \lambda \in \Lambda \text{ and } \lambda_0 t = t\} \tag{2.1}$$

where Λ denotes the space of strictly increasing, continuous functions from $[0, 1]$ onto $[0, 1]$. Detailed geometric and probabilistic properties of the space D with the uniform norm and the Skorohod metric can be found in Billingsley (1968), pp. 109–153 and Taylor (1978), pp. 153–184. However, a few properties will be discussed to illustrate the troublesome aspects of D and the need for particular techniques and hypotheses. In particular, the relation

$$d(x + u, y + v) \leqslant d(x, y) + \| u \|_\infty + \| v \|_\infty \tag{2.2}$$

will be frequently used.

The Banach space obtained by giving D the uniform topology is not separable. Not only does the nonseparability prohibit applying many known convergent results, it also restricts the variety of random elements for D. With the Skorohod metric topology, the linear space is separable but not a linear topological space (addition is not continuous). However, the Skorohod topology does provide an ideal Borel structure for the natural consideration of random elements in D.

A random element in D is a function $X : \Omega \to D$ from a probability space (Ω, F, P) into D which is measurable with respect to the Skorohod Borel subsets of D. Thus, X is a random element in D if and only if $X(t)$ is a random variable for each $t \in [0, 1]$, Billingsley (1968), p. 128. If $E \| X \|_\infty < \infty$, then the expected value can be defined pointwise by $(EX)(t) = E[X(t)]$ for each $t \in [0, 1]$, and $EX \in D$.

The absence of translation invariance and local convexity necessitates the following modification of the concept of tightness in order to develope stochastic convergence results in D.

Definition. A sequence of random elements $\{X_n\}$ in D is said to be *convex* *tight* if for each $\epsilon > 0$ there exists a convex, compact set K_ϵ such that $P[X_n \in K_\epsilon] > 1 - \epsilon$ for all n.

Characteristics of convex tightness and the limitations of this concept are described in Section 7.4 of Taylor (1978) and by Daffer (to appear).

For each positive integer m, let $E_{i,m} = [i/2^m, (i+1)/2^m)$ for $i = 0, 1, \ldots, 2^{m-1}$ and let $E_{2^m,m} = \{1\}$. Define the linear Borel measurable function $T_m : D \to D$ by

$$T_m(x) = \sum_{i=0}^{2^m} x\left(\frac{i}{2^m}\right) I_{E_{i,m}} \tag{2.3}$$

For each $x \in D$ $\lim_{m\to\infty} d(x, T_m(x)) = 0$, and $\lim_{m\to\infty} \sup_{x\in K} d(x, T_m x) = 0$ when K is a (Skorohod) compact set. Finally, let $\{a_{nk}\}$ denote a Toeplitz (double) sequence of scalars, that is,

$$\lim_{n\to\infty} a_{nk} = 0 \qquad \text{for each } k = 1, 2, \ldots \tag{2.4}$$

and

$$\sum_{k=1}^{\infty} | a_{nk} | \leqslant C \qquad \text{for each } n \tag{2.5}$$

and for some constant C which can be assumed to be 1.

3. Convergence in Probability

The results in this section are based on the real-valued random variable results of Pruitt (1966) and Rohatgi (1971) and on the Banach space random element results of Wei and Taylor (1978). While the Skorohod metric provides the desired natural setting, results will be stated for convergence in the stronger uniform norm when possible.

For a sequence $\{X_n\}$ of random elements, the condition (CT) is defined as:

(CT) for each $\epsilon > 0$ there exists K_ϵ, convex and compact,

such that $E \| X_n I_{[X_n \notin K]} \|_\infty < \epsilon$ for all n.

Note that convex tightness and uniformly bounded pth moments ($p > 1$) implies (CT) and that (CT) implies the existence of EX.

Theorem 1. *Let $\{X_n\}$ be a sequence of random elements in D which satisfies*

(CT) and let $\{a_{nk}\}$ be a double array of real numbers satisfying $\sum_{k=1}^{\infty} |a_{nk}| \leqslant 1$ for each n. If

$$\left| \sum_{k=1}^{n} a_{nk}(X_k(t) - EX_k(t)) \right| \to \textit{in probability}$$

for each dyadic rational in $[0, 1]$, *then*

$$d\left(\sum_{k=1}^{n} a_{nk} X_k, \sum_{k=1}^{n} a_{nk} EX_k\right) \to 0 \textit{ in probability.}$$

Proof. Let $0 < \epsilon < 1$ and $0 < \delta < 1$ be given. By (CT) choose K such that

$$E \| X_n I_{[X_n \notin K]} \| \leqslant \frac{\epsilon\delta}{24} \qquad \text{for all } n \tag{3.1}$$

and such that $0 \in K$(w.l.o.g). Next, pick m such that $d(x, T_m x) < \epsilon/6$ for all $x \in K$. For each n

$$\begin{aligned} P&\left[d\left(\sum_{k=1}^{n} a_{nk} X_k, \sum_{k=1}^{n} a_{nk} EX_k\right) > \epsilon\right] \\ &\leqslant P\left[d\left(\sum_{k=1}^{n} a_{nk} X_k, \sum_{k=1}^{n} a_{nk} T_m X_k\right) > \frac{\epsilon}{3}\right] \\ &\quad + P\left[d\left(\sum_{k=1}^{n} a_{nk} T_m X_k, \sum_{k=1}^{n} a_{nk} T_m(EX_k)\right) > \frac{\epsilon}{3}\right] \\ &\quad + P\left[d\left(\sum_{k=1}^{n} a_{nk} T_m(EX_k), \sum_{k=1}^{n} a_{nk} EX_k\right) > \frac{\epsilon}{3}\right]. \end{aligned} \tag{3.2}$$

Using (CT) and (2.2), the first term of (3.2) can be expressed as

$$\begin{aligned} P&\left[d\left(\sum_{k=1}^{n} a_{nk} X_k, \sum_{k=1}^{n} a_{nk} T_m X_k\right) > \frac{\epsilon}{3}\right] \\ &\leqslant P\left[\left\| \sum_{k=1}^{n} a_{nk} X_k I_{[X_k \notin K]} \right\|_\infty + \left\| \sum_{k=1}^{n} a_{nk} T_m X_k I_{[X_k \notin K]} \right\|_\infty > \frac{\epsilon}{6}\right] \\ &\leqslant \frac{12}{\epsilon} \sum_{k=1}^{n} |a_{nk}| \, E \| X_k I_{[X_k \notin K]} \|_\infty < \frac{\delta}{2}. \end{aligned} \tag{3.3}$$

Since K is convex, compact (and balanced if necessary), and $0 \in K$, then

$$\sum_{k=1}^{n} a_{nk} X_k I_{[X_k \in K]} \in K \qquad \text{and} \qquad \sum_{k=1}^{n} a_{nk} E(X_k I_{[X_k \in K]}) \in K.$$

Thus, the third term of (3.2) can be expressed as

$$P\left[d\left(\sum_{k=1}^{n} a_{nk} T_m(EX_k), \sum_{k=1}^{n} a_{nk} EX_k\right) > \frac{\epsilon}{3}\right]$$

$$\leqslant P\left[\left\| \sum_{k=1}^{n} a_{nk} T_m(EX_k I_{[X_k \notin K]}) \right\|_{\infty} + \left\| \sum_{k=1}^{n} a_{nk} EX_k I_{[X_k \notin K]} \right\|_{\infty} > \frac{\epsilon}{6}\right].$$

Hence, the third term of (3.2) has probability 0 since $\| T_m x \|_\infty \leqslant \| x \|_\infty$ and (3.1) holds. For the second term of (3.2),

$$P\left[d\left(\sum_{k=1}^{n} a_{nk} T_m X_k, \sum_{k=1}^{n} a_{nk} T_m(EX_k)\right) > \frac{\epsilon}{3}\right]$$

$$\leqslant P\left[\left\| \sum_{k=1}^{n} a_{nk} T_m(X_k - EX_k) \right\|_{\infty} > \frac{\epsilon}{3}\right]$$

$$\sum_{i=0}^{2^m} P\left[\left| \sum_{k=1}^{n} a_{nk} \left(X_k\left(\frac{i}{2m}\right) - EX_k\left(\frac{i}{2^m}\right)\right) \right| > \frac{\epsilon}{3(2^m+1)}\right]$$

$$< \frac{\delta}{2} \qquad \text{for} \quad n \geqslant N \tag{3.4}$$

from the hypothesis of convergence in probability for each dyadic rational. From (3.3) and (3.4), it follows that

$$P\left[d\left(\sum_{k=1}^{n} a_{nk} X_k, \sum_{k=1}^{n} a_{nk} EX_k\right) > \epsilon\right] < \delta \qquad \text{for} \quad n \geqslant N. \quad \blacksquare$$

If $\sum_{k=1}^{n} a_{nk} EX_k$ converges to a constant, then Theorem 1 is "if and only if" since convergence in probability implies weak convergence (in distribution) and hence pointwise convergence in distribution to a constant which yields pointwise convergence in probability. If the random elements $\{X_n\}$ are identically distributed, then a first moment condition and convex tightness suffices for (CT). Also, the condition that $\max_{1 \leqslant k \leqslant n} | a_{nk} | \to 0$ and the condition of pointwise uncorrelation or pointwise independence provide for the pointwise convergence in probability of the hypothesis in Theorem 1. Thus, Theorem 1 provides more applicable results for the subspace $C[0, 1]$ of $D[0, 1]$ than the results of Wei and Taylor (1978) since the concepts of convex tightness and tightness coincide for Banach spaces.

A weak law is available for weighted sums of independent random elements which satisfy (CT). This result will use truncation techniques from the next section.

THEOREM 2. *Let $\{X_k\}$ be a sequence of independent random elements in D satisfying (CT) and $EX_k = 0$ for all k. If $\{a_{nk}\}$ is a Toeplitz sequence satisfying* $\max_{1\leqslant k\leqslant n} | a_{nk} | = O(n^{-\alpha})$ *for some $\alpha > 0$, then*

$$\lim_{n\to\infty} E\left\| \sum_{k=1}^{n} a_{nk}X_k \right\| = 0.$$

Proof. Let $\epsilon > 0$ be given and K, convex and compact, be chosen according to (CT). Write

$$E\left\| \sum_{k=1}^{n} a_{nk}X_k \right\|_\infty \leqslant E\left\| \sum_{k=1}^{n} a_{nk}X_k I_{[X_k\in K]} \right\|_\infty + E\left\| \sum_{k=1}^{n} a_{nk}X_k I_{[X_k\notin K]} \right\|_\infty . \tag{3.5}$$

Now

$$E\left\| \sum_{k=1}^{n} a_{nk}X_k I_{[X_k\notin K]} \right\|_\infty \leqslant \sum_{k=1}^{n} | a_{nk} | E \| X_k I_{[X_k\notin K]} \|_\infty \leqslant \sum_{k=1}^{n} | a_{nk} | \epsilon \leqslant \epsilon \tag{3.6}$$

by (CT).

Write the other term as

$$E\left\| \sum_{k=1}^{n} a_{nk}X_k I_{[X_k\in K]} \right\|_\infty \leqslant E\left\| \sum_{k=1}^{n} a_{nk}(X_k I_{[X_k\in K]} - E(X_k I_{[X_k\in K]})) \right\|_\infty$$
$$+ E\left\| \sum_{k=1}^{n} a_{nk} E(X_k I_{[X_k\in K]}) \right\|_\infty .$$

Now

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} a_{nk}(X_k I_{[X_k\in K]} - E(X_k I_{[X_k\in K]})) \right\|_\infty = 0, \qquad \text{a.s.},$$

follows by the hypothesis on $\{a_{nk}\}$ immediately from Theorem 3 of the next section since the convex hull of $K + K$ is still conditionally compact. Next,

$$\left\| \sum_{k=1}^{n} a_{nk}(X_k I_{[X_k\in K]} - E(X_k I_{[X_k\in K]})) \right\|_\infty$$
$$\leqslant \sum_{k=1}^{n} | a_{nk} | (\| X_k I_{[X_k\in K]} \|_\infty + E \| X_k I_{[X_k\in K]} \|_\infty)$$
$$\leqslant \sum_{k=1}^{n} | a_{nk} | 2A \leqslant 2A, \qquad \text{where} \quad A = \sup_{x\in K} \| x \|_\infty .$$

Thus by the dominated convergence theorem,

$$\lim_{n\to\infty} E\left\| \sum_{k=1}^{n} a_{nk}(X_k I_{[X_k\in K]} - E(X_k I_{[X_k\in K]}))\right\|_\infty = 0. \tag{3.7}$$

Finally, note that $E(X_k I_{[X_k\in K]}) = -E(X_k I_{[X_k\in K]})$, and so

$$\begin{aligned} E\left\| \sum_{k=1}^{n} a_{nk} E(X_k I_{[X_k\in K]})\right\|_\infty &= \left\| \sum_{k=1}^{n} a_{nk} E(X_k I_{[X_k\notin K]})\right\|_\infty \\ &\leqslant \sum_{k=1}^{n} | a_{nk} | \, E \| X_k I_{[X_k\notin K]} \|_\infty \\ &\leqslant \sum_{k=1}^{n} | a_{nk} | \, \epsilon \leqslant \epsilon. \end{aligned} \tag{3.8}$$

Now (3.5), (3.6), (3.7), and (3.8) together give

$$\lim_{n\to\infty} E\left\| \sum_{k=1}^{n} a_{nk} X_k \right\|_\infty \leqslant \epsilon + \epsilon = 2\epsilon. \qquad \text{Since } \epsilon \text{ is arbitrary,}$$

$$\lim_{n\to\infty} E\left\| \sum_{k=1}^{n} a_{nk} X_k \right\| = 0. \quad \blacksquare$$

The hypotheses of Theorem 2 do not imply the strong law of large numbers (rf. Hoffmann–Jørgensen and Pisier (1976), remark, p. 592); however, the weak law of large numbers follows since convergence in the mean implies convergence in probability.

4. Almost Sure Convergence

Almost sure convergence of the weighted sums is obtained for independent, convex tight random elements and for independent, identically distributed random elements in this section. The convex tight results will be obtained in two parts. First, the result will be obtained when the random elements are restricted to a convex, compact set. Truncation to the convex, compact set will be used for the general result.

THEOREM 3. *Let K be a convex, compact subset of D. Let $\{X_n\}$ be independent random elements with $EX_n = 0$ and $P[X_n \in K] = 1$ for all n. If $\{a_{nk}\}$ is a Toeplitz sequence such that* $\max_{1 \leqslant k \leqslant n} | a_{nk} | = O(n^{-\alpha})$ *for some $\alpha > 0$, then*

$$\left\| \sum_{k=1}^{n} a_{nk} X_k \right\| \to 0 \qquad \textit{almost surely.}$$

Proof. First, it can be assumed that K is symmetric. For each $t \in [0, 1]$, $| X(t)| \leqslant \sup_{x \in K} \| x \|_\infty < \infty$ almost surely since K is compact. By Theorem 2 of Rohatigi (1971),

$$\left| \sum_{k=1}^{n} a_{nk} X_k(t) \right| \to 0 \qquad \text{almost surely} \tag{4.1}$$

for each rational $t \in [0, 1]$ since $\{X_k(t)\}$ are independent, uniformly bounded (a.s) random variables with zero means. Since K is convex and symmetric, $\sum_{k=1}^{n} a_{nk} X_k \in K$ almost surely. Thus

$$\left\| \sum_{k=1}^{n} a_{nk} X_k \right\|_\infty \to 0 \qquad \text{almost surely}$$

by taking the countable union of null sets in (4.1). ∎

THEOREM 4. *Let $\{X_n\}$ be a sequence of convex tight, independent random elements in D satisfying $EX_n = 0$ and $E[\| X_n \|^r] \leqslant \Gamma$ for all n where $r > 1$. Let $\{a_{nk}\}$ be a Toeplitz sequence such that* $\max_{1 \leqslant k \leqslant n} | a_{nk} | = O(n^{-s})$ *for some $0 < 1/s < r - 1$. Then,*

$$\left\| \sum_{k=1}^{n} a_{nk} X_k \right\|_\infty \to 0 \qquad \textit{almost surely.}$$

Proof. First, Γ can be assumed to be 1. Given $\epsilon > 0$, let

$$\delta = (\epsilon/4)^{r/(r-1)}. \tag{4.2}$$

Let K be compact, convex and symmetric and chosen so that

$$P[X_n \in K] > 1 - \delta. \tag{4.3}$$

Define

$$Y_n = X_n I_{[X_n \in K]} \qquad \text{and} \qquad Z_n = X_n - Y_n .$$

Then $\{Y_n - EY_n\}$ takes values in $K + K$, and hence

$$\left\| \sum_{k=1}^{n} a_{nk}(Y_k - EY_k) \right\| \to 0 \tag{4.3}$$

almost surely by Theorem 3. By Hölder's inequality

$$\begin{aligned} E \| Z_n \| &= E(\| X_n \|_\infty I_{[X_n \notin K]}) \\ &\leqslant (E \| X_n \|_\infty^r)^r (P[X_n \notin K])^{(r-1)/r} \\ &\leqslant 1(\delta^{(r-1)/r}) = \epsilon/4 \qquad \text{for each } n. \end{aligned} \tag{4.4}$$

Also, for each n

$$E \mid \| Z_n \| - E \| Z_n \| \mid^r \leqslant 2^r. \tag{4.5}$$

Dominance in probability of Rohatgi (1971) follows from (4.5), and hence

$$\sum_{k=1}^{n} | a_{nk} | (\| Z_k \| - E \| Z_k \|) \to 0 \tag{4.6}$$

almost surely. Since a sequence of $\epsilon_n \to 0$, $\delta_n \to 0$, and corresponding compact sets K_n could be chosen, a countable number of null sets can be excluded in (4.3) and (4.6). Thus, for almost all $\omega \in \Omega$, there is an $N(\omega, \epsilon)$ such that for $n \geqslant N(\omega, \epsilon)$,

$$\left\| \sum_{k=1}^{n} a_{nk}(Y_k(\omega) - EY_k) \right\| < \epsilon/4 \tag{4.7}$$

and

$$\left\| \sum_{k=1}^{n} a_{nk}(\| Z_k(\omega) \| - E \| Z_k \|) \right\| < \epsilon/4. \tag{4.8}$$

From (4.4) and (4.8), it follows that

$$\begin{aligned} \left\| \sum_{k=1}^{n} a_{nk} Z_k(\omega) \right\| &\leqslant \sum_{k=1}^{n} | a_{nk} | (\| Z_k(\omega) \| - E \| Z_k \|) \\ &\quad + \sum_{k=1}^{n} | a_{nk} | E \| Z_k \| \\ &\leqslant \epsilon/4 + \epsilon/4 = \epsilon/2 \end{aligned} \tag{4.9}$$

for $n \geqslant N(\epsilon, \omega)$. Similarly, $EY_k = -EZ_k$, (4.4), and (4.7) yields

$$\left\| \sum_{k=1}^{n} a_{nk} Y_k(\omega) \right\|. < \epsilon/2 \tag{4.10}$$

for all $n \geqslant N(\omega, \epsilon)$. Thus, from (4.9) and (4.10)

$$\left\| \sum_{k=1}^{n} a_{nk} X_k(\omega) \right\| < \epsilon$$

for all $n \geqslant N(\epsilon, \omega)$. ∎

Tightness results in a complete metric space incorporate the identical distribution results since each probability measure is tight. However, identically distributed random elements in D need not be convex tight and certainly not conversely. Thus, the last result will be the almost sure convergence of weighted sums of independent, identically distributed random elements. The result is based on Ranga Rao's (1963) strong law of large number.

LEMMA 5. [Ranga Rao (1963)]. *Let X be a random element in D with $E\| X \|_\infty < \infty$. Then, for every $\epsilon > 0$, there is a partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ of $[0, 1]$ such that* $\max_i \sup_{s,t \in J_i} E \mid X(s) - X(t) \mid \leqslant \epsilon$, *where $J_i = [t_{i-1}, t_i)$.*

The following property of compact sets in D is needed; if K is compact in D, then to any $\epsilon > 0$ there is $\delta > 0$ such that $| x(t) - x(\alpha)| \leqslant | x(\beta - 0) - x(\alpha)| + \epsilon$, whenever $\alpha \leqslant t < \beta < \alpha + \delta$. This follows from the characterization of compact sets in D in terms of $w_x''(\delta)$ (Billingsley (1968), pp. 118–120).

THEOREM 6. *Let $\{X_n\}$ be a sequence of independent, identically distributed random elements in D and let $\{a_{nk}\}$ be a Toeplitz sequence such that* $\max_{1 \leqslant k \leqslant n} | a_{nk} | = O(n^{-\alpha})$. *If $E \| X_1 \|_\infty^{1+1/\alpha} < \infty$, then*

$$\left\| \sum_{k=1}^{n} a_{nk}(X_k - EX_1) \right\| \to 0 \qquad \text{almost surely.}$$

Proof. Let $\epsilon > 0$ be given, and let a compact subset K of D be chosen such that $E[\| X_1 \| I_{[X_1 \notin K]}] \leqslant \epsilon$. The choice of K is possible since the probability measure of X_1 is tight. Now let $\delta > 0$ be chosen as above so that $\alpha \leqslant t < \beta < \alpha + \delta$ implies $| x(t) - x(\alpha)| \leqslant | x(\beta - 0) - x(\alpha)| + \epsilon$, for any $x \in K$. By Lemma 5 choose a partition $\{t_i\}_{i=0}^{N}$ of $[0, 1]$ satisfying

$$\max_{i=0,1,\ldots,N} \sup_{s,t \in J_i} E \mid X_1(t) - X_1(s) \mid \leqslant \epsilon, \qquad \text{where} \quad J_i = [t_{i-1}, t_i)$$

and also satisfying $t_i - t_{i-1} < \delta$, for $i = 0, 1, \ldots, N$.

Thus,

$$\left| \sum_{k=1}^{n} a_{nk}(X_k(t) - EX_1(t)) \right|$$

$$\leqslant \left| \sum_{k=1}^{n} a_{nk}(X_k(t)\, I_{[X_k \in K]} - EX_1(t)\, I_{[X_k \in K]}) \right| + \left\| \sum_{k=1}^{n} a_{nk} X_k I_{[X_k \notin K]} \right\|_\infty$$

$$+ E[\| X_1 \|_\infty I_{[X_1 \notin K]}]$$

$$\leqslant \left| \sum_{k=1}^{n} a_{nk}(X_k(t)\, I_{[X_k \in K]} - E(X_1(t)\, I_{[X_k \in K]})) \right|$$

$$+ \sum_{k=1}^{n} | a_{nk} |\, \| X_k \|_\infty I_{[X_k \notin K]} + E[\| X_k \|_\infty I_{[X_k \notin K]}] \tag{4.11}$$

for each $t \in [0, 1]$. Since $\{\| X_k \|_\infty I_{[X_k \notin K]}\}$ are independent, identically distributed random variables with

$$E(\| X_k \|_\infty I_{[X_k \notin K]})^{1+1/\alpha} \leqslant E(\| X_k \|_\infty^{1+1/\alpha}) < \infty, \qquad \text{then}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} | a_{nk} |\, \| X_k \|_\infty I_{[X_k \notin K]} \overset{\text{a.s.}}{=} E[\| X_k \|_\infty I_{[X_k \notin K]}]. \tag{4.12}$$

For the first term of (4.11)

$$\sup_{t \in J_i} \left| \sum_{k=1}^{n} a_{nk}(X_k(t)\, I_{[X_k \in K]} - EX_1(t)\, I_{[X_k \in K]}) \right|$$

$$\leqslant \left| \sum_{k=1}^{n} a_{nk}(X_k(t_{i-1})\, I_{[X_k \in K]} - EX_1(t_{i-1})\, I_{[X_1 \in K]}) \right|$$

$$+ \sup_{t \in J_i} \left| \sum_{k=1}^{n} a_{nk} X_k(t)\, I_{[X_k \in K]} - \sum_{k=1}^{n} a_{nk} X_k(t_{i-1})\, I_{[X_k \in K]}) \right|$$

$$+ \sup_{t \in J_i} | EX_1(t)\, I_{[X_1 \in K]} - EX_1(t_{i-1})\, I_{[X_1 \in K]} |$$

$$\leqslant \left| \sum_{k=1}^{n} a_{nk}(X_k(t_{k-1})\, I_{[X_k \in K]} - EX_1(t_{i-1})\, I_{[X_1 \in K]}) \right|$$

$$+ \sum_{k=1}^{n} | a_{nk} |\, |[X_k(t_i - 0) - X_k(t_{i-1})]\, I_{[X_k \in K]} |$$

$$+ E(| X_1(t_i - 0) - X_1(t_{i-1})|\, I_{[X_1 \in K]}) + 2\epsilon, \quad \text{for} \quad i = 0, \ldots, N \tag{4.13}$$

by compactness of K and using $t_i - t_{i-1} < \delta$. Similar to (4.12)

$$\lim_{n\to\infty} \left| \sum_{k=1}^{n} a_{nk}(X_k(t_{i-1}) I_{[X_k \in K]} - E(X_1(t_{i-1}) I_{[X_1 \in K]})) \right| \overset{\text{a.s.}}{=} 0 \tag{4.14}$$

for each $i = 0, \ldots, N$.

From (4.11), (4.13), and (4.14)

$$\begin{aligned} &\limsup_{n\to\infty} \left\| \sum_{k=1}^{n} a_{nk}(X_k I_{[X_k \in K]} - E(X_1 I_{[X_1 \in K]})) \right\|_\infty \\ &\quad \leqslant 2 \max_{0 \leqslant i \leqslant N} E[| X_1(t_i - 0) - X_1(t_{i-1})| I_{[X_1 \in K]}] + 2\epsilon \\ &\quad \leqslant 2\epsilon + 2\epsilon = 4\epsilon \qquad \text{a.s..} \end{aligned} \tag{4.15}$$

Finally, using (4.11), (4.12), and (4.15)

$$\limsup_{n\to\infty} \left\| \sum_{k=1}^{n} a_{nk} X_k - EX_1 \right\|_\infty \leqslant 2E(\| X_1 \|_\infty I_{[X_1 \in K]}) + 4\epsilon \leqslant 6\epsilon \qquad \text{a.s.,}$$

and the proof is completed since ϵ is arbitrary. ∎

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